P246 BEHAVIOUR OF QUASI_SHEAR WAVES IN ORTHORHOMBIC MEDIA

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Summary

The media with an orthorhombic symmetry typically have a finite number of directions in which the velocities of two quasi-shear waves are the same. From now on, the directions are referred to as singular.

For such class of media, the sheets of quasi-shear wave velocitiy surface in the neighbourhood of the singular directions intersect on a conic surface. The shear wave polarization vectors turn about when rounding singular direction.

Introduction

Shoenberg M. and Helbig K. (1997) have given an estimation of the number of possible singular directions and have determined their positions in space for the orthorhombic media. They also have noted that the singular directions are characterized by fast changes of both the polarization vector and the shear wave velocity with respect to changes in the direction of wave propagation.

The given paper is devoted to studying the structure of the quasi-shear wave velocity surface being formed by the function $V_S(\alpha,\varphi)$ with α being the phase angle with vertical axis and φ being an azimuth. Furthermore, the directions of polarization vectors in the neighbourhood of a singular direction are also investigated.

Let us choose a system of coordinates so that the planes of an orthorhombic symmetry coincide with coordinate planes $[x_1, x_2]$, $[x_1, x_3]$,and $[x_2, x_3]$. Then we denote the density-normalized stiffness in this system of coordinates as c_{ij}.

According to the results by Shoenberg M. and Helbig K., the singular directions can be divided on 3 categories.

1. A wave propagates along one of the coordinate axes x_1, x_2, x_3. This is the case when one of the following equalities is met: $c_{55}=c_{66}$, $c_{44}=c_{66}$, or $c_{44}=c_{55}$. We shall not discuss these cases further.
2. A wave propagates along one of the coordinate planes. In this case, the angle α_0 can be determined by solving a square equation in the unknown $\tan^2(\alpha_0)$. For example, in the plane $[x_1, x_3]$, such equation is

$$\begin{aligned}&(c_{55}-c_{66})(c_{55}-c_{66})\tan^4(\alpha_0)+\\&[(c_{33}-c_{44})(c_{11}-c_{66})+(c_{55}-c_{44})(c_{55}-c_{66})-\\&(c_{13}+c_{55})^2]\tan^2(\alpha_0)+(c_{33}-c_{44})(c_{55}-c_{44})=0\end{aligned} \quad (1)$$

3. The direction of wave propagation is off any coordinate plane. In this case, for finding (α_0, φ_0) it is necessary to solve the linear system of three equations in the unknowns s_1^2, s_2^2, and s_3^2:

$$\begin{cases}\left(c_{11}-\dfrac{(c_{12}+c_{66})(c_{13}+c_{55})}{c_{23}+c_{44}}\right)s_1^2+c_{66}s_2^2+c_{55}s_3^2=1\\ c_{66}s_1^2+\left(c_{22}-\dfrac{(c_{23}+c_{44})(c_{12}+c_{66})}{c_{13}+c_{55}}\right)s_2^2+c_{44}s_3^2=1\\ c_{55}s_1^2+c_{44}s_2^2+\left(c_{33}-\dfrac{(c_{13}+c_{55})(c_{23}+c_{44})}{c_{12}+c_{66}}\right)s_3^2=1\end{cases},(2)$$

and then to find the angles using the formulas

$$\alpha_0 = \cos^{-1}\left(\frac{s_3}{s}\right),\ \varphi_0 = \cos^{-1}\left(\frac{s_1}{s \cdot \sin(\alpha)}\right),$$
$$s = \sqrt{s_1^2 + s_2^2 + s_3^2}$$

In this paper, we consider the case when the above equations have different roots in the unknowns α_0 and φ_0. The case of repeated roots was investigated by Crampin S. and Yedlin M. (1981).

The squared phase velocities V^2 are eigenvalues, and polarization vectors $\mathbf{p} = (p_i)$ are eigenvectors of the symmetric matrix $\mathbf{G}(\alpha,\varphi) = (g_{ij})$ with the following elements:

$$\begin{aligned} g_{11} &= c_{11}n_1^2 + c_{66}n_2^2 + c_{55}n_3^2 \\ g_{22} &= c_{66}n_1^2 + c_{22}n_2^2 + c_{44}n_3^2 \\ g_{33} &= c_{55}n_1^2 + c_{44}n_2^2 + c_{33}n_3^2 \\ g_{12} &= (c_{12} + c_{66})n_1 n_2 \\ g_{13} &= (c_{13} + c_{55})n_1 n_3 \\ g_{23} &= (c_{23} + c_{44})n_2 n_3 \end{aligned} \quad , \qquad (3)$$

where

$$n_1 = \cos\varphi \sin\alpha, \quad n_2 = \sin\varphi \sin\alpha, \quad n_3 = \cos\alpha$$

are the components of a directing vector $\mathbf{n} = (n_j)$.

Let $V_S^2(\alpha_0,\varphi_0)$ and $V_P^2(\alpha_0,\varphi_0)$ be eigenvalues of a matrix $\mathbf{G}(\alpha_0,\varphi_0)$, first of which is double. Assume that $\mathbf{P} = \{\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3\}$ is a basis formed by the related normalized eigenvectors.

Thus, the study of both $V_S^2(\alpha,\varphi)$ and $\mathbf{p}_i(\alpha,\varphi)$ $(i = 1,2)$ in the neighbourhood of a singular direction can be reduced to the process of estimating the perturbated eigenvalues and eigenvectors of the symmetric matrix $\mathbf{G}(\alpha_0,\varphi_0)$, having the double eigenvalue.

Method

Consider a perturbation of the matrix $\mathbf{G}(\alpha_0,\varphi_0)$ in the basis **P**. Let us designate the matrix of transforming to this basis as $\mathbf{S} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$. If $\alpha = \alpha_0 + d\alpha$ and $\varphi = \varphi_0 + d\varphi$, then

$$\mathbf{S}^T\mathbf{G}(\alpha,\varphi)\mathbf{S} = \begin{bmatrix} V_S^2(\alpha_0,\varphi_0) & 0 & 0 \\ 0 & V_S^2(\alpha_0,\varphi_0) & 0 \\ 0 & 0 & V_P^2(\alpha_0,\varphi_0) \end{bmatrix} +$$
$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12} & L_{22} & L_{23} \\ L_{13} & L_{23} & L_{33} \end{bmatrix} + o(d\alpha, d\varphi). \qquad (4)$$

In (4), the matrix $\mathbf{L} = (L_{ij})$ has elements that are linear forms of the variables $d\alpha$ and $d\varphi$:

$$\mathbf{L} = \mathbf{S}^T\left(\frac{\partial \mathbf{G}}{\partial \alpha}d\alpha + \frac{\partial \mathbf{G}}{\partial \varphi}d\varphi\right)\mathbf{S} \cdot$$

The particular derivatives of the matrix **G** are taken for each element at the point (α_0,φ_0).

Let us define as

$$\mathbf{M} = \begin{bmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{bmatrix}$$

the left top 2x2-minor of the matrix **L**.

According to the theory of perturbation of double eigenvalues of matrixes (Lankaster, 1969), with the accuracy of $o(d\alpha, d\varphi)$, we can find the eigenvalues and eigenvectors of the matrix $\mathbf{G}(\alpha,\varphi)$ in the basis **P** from the equations

$$V_S^2(\alpha,\varphi) = V_S^2(\alpha_0,\varphi_0) + \lambda$$

and

$$\mathbf{p}_S(\alpha,\varphi) = \begin{bmatrix} L_{12} & \lambda - L_{11} & 0 \end{bmatrix}^T$$

where

$$\lambda = \frac{1}{2}(L_{11} + L_{22}) \pm \frac{1}{2}\sqrt{(L_{11} - L_{22})^2 + 4L_{12}^2}\,. \qquad (5)$$

Let us define as ψ the angle that satisfies $\tan\psi = \dfrac{2L_{12}}{L_{11} - L_{22}}$, and find the angle ψ_S between the vectors $\mathbf{p}_S(\alpha,\varphi)$ and $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$.

Since

$$\tan\psi_S = \frac{\lambda - L_{11}}{L_{12}},$$

then

$$\tan 2\psi_S = \frac{2\tan\psi_S}{1 - \tan^2\psi_S} = \frac{2L_{12}}{L_{11} - L_{22}} = \tan\psi.$$

For a typical medium with an orthorhombic symmetry the forms $L_{11} - L_{22}$ and L_{12} are not proportional. In this case, the vector $\mathbf{v} = \begin{bmatrix} d\alpha & d\varphi \end{bmatrix}^T$ can be presented as $\mathbf{w} = \mathbf{A}\mathbf{v}$, where **A** is a linear non-singular mapping. Therefore, the vector $\mathbf{w} = \begin{bmatrix} L_{11} - L_{22} & 2L_{12} \end{bmatrix}^T$ turns on 360^0, when the vector **v** rotates over an ellipse. At the same time, the vector $\mathbf{p}_S(\alpha,\varphi)$ turns on 180^0, running over a circle.

Taking into account (5), we find that

$$V_S(\alpha,\varphi) = V_S(\alpha_0,\varphi_0) + \frac{1}{4V_S(\alpha_0,\varphi_0)}\left[(L_{11} + L_{22}) \pm \sqrt{(L_{11} - L_{22})^2 + 4L_{12}^2}\right]. \qquad (6)$$

is true with accuracy of $o(d\alpha, d\varphi)$.

The linear part of the perturbation

$$\frac{L_{11} + L_{22}}{4V_S(\alpha_0,\varphi_0)}$$

is identical for both shear waves. The term

$$\pm\frac{\sqrt{(L_{11} - L_{22})^2 + 4L_{12}^2}}{4V_S(\alpha_0,\varphi_0)}$$

defines a conic elliptic surface and its different signs correspond to different sheets of velocity surface.

Consider in more detail the case of the location of a singular direction in the plane [x_1, x_3]. Then $\varphi_0 = 0$ and the angle α_0 can be obtained using (1). Matrixes $\mathbf{G}(\alpha_0,\varphi_0)$ and

$$\mathbf{H} = \frac{\partial\mathbf{G}}{\partial\alpha}d\alpha + \frac{\partial\mathbf{G}}{\partial\varphi}d\varphi$$

have respectively the forms:

$$\mathbf{G}(\alpha_0,\varphi_0)=\begin{bmatrix} g_{11} & 0 & g_{13} \\ 0 & g_{22} & 0 \\ g_{13} & 0 & g_{33} \end{bmatrix},$$

and

$$\mathbf{H}=\begin{bmatrix} h_{11}\,d\alpha & h_{12}\,d\varphi & h_{13}\,d\alpha \\ h_{12}\,d\varphi & h_{22}\,d\alpha & h_{23}\,d\varphi \\ h_{13}\,d\alpha & h_{23}\,d\varphi & h_{33}\,d\alpha \end{bmatrix}.$$

The element g_{22} is the double eigenvalue of the matrix $\mathbf{G}(\alpha_0,\varphi_0)$. Therefor

$$(g_{11}-g_{22})(g_{33}-g_{22})=g_{13}^2.$$

The values of g_{ij} are defined from (3), in which

$$n_1=\sin\alpha_0,\ n_2=0,\ n_3=\cos\alpha_0,$$

and the values of h_{ij} are defined as:

$$\begin{aligned} h_{11}&=(c_{11}-c_{55})\sin 2\alpha_0 \\ h_{12}&=(c_{12}+c_{66})\sin^2\alpha_0, \\ h_{13}&=(c_{13}+c_{55})\cos 2\alpha_0, \\ h_{22}&=(c_{66}-c_{44})\sin 2\alpha_0, \\ h_{23}&=(c_{23}+c_{44})\sin\alpha_0\cos\alpha_0, \\ h_{33}&=(c_{55}-c_{33})\sin 2\alpha_0. \end{aligned}$$

Let us designate

$$r_0=g_{11}+g_{33}-2g_{22},\ r_1=(g_{33}-g_{22})h_{11}+(g_{11}-g_{22})h_{33}-2g_{13}\,h_{13},$$
$$r_2=(g_{33}-g_{22})h_{12}^2+(g_{11}-g_{22})h_{23}^2-2g_{13}\,h_{12}\,h_{23}.$$

Then the normalized eigenvectors of the matrix $\mathbf{G}(\alpha_0,\varphi_0)$ can be found with the formulas

$$\mathbf{p}_1=\frac{1}{r}\begin{bmatrix} g_{13} \\ 0 \\ g_{22}-g_{11} \end{bmatrix},\quad \mathbf{p}_2=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\quad \mathbf{p}_3=\frac{1}{r}\begin{bmatrix} g_{11}-g_{22} \\ 0 \\ g_{13} \end{bmatrix}$$

where

$$r^2=(g_{11}-g_{22})r_0.$$

With the help of algebraic transformations, we can obtain

$$L_{11}=\mathbf{p}_1^T\mathbf{H}\mathbf{p}_1=\frac{r_1}{r_0}d\alpha,$$

$$L_{12}^2=\left(\mathbf{p}_1^T\mathbf{H}\mathbf{p}_2\right)^2=\frac{r_2}{r_0}d\varphi^2,$$

$$L_{22}=\mathbf{p}_2^T\mathbf{H}\mathbf{p}_2=h_{22}\,d\alpha.$$

Therefore,

$$V_S(\alpha,\varphi) = V_S(\alpha_0,\varphi_0) + \frac{1}{4V_S(\alpha_0,\varphi_0)}\left[\left(\frac{r_1}{r_0} + h_{22}\right)d\alpha \pm \sqrt{\left(\frac{r_1}{r_0} - h_{22}\right)^2 d\alpha^2 + 4\frac{r_2}{r_0}d\varphi^2}\right] \tag{7}$$

and

$$\tan 2\psi_S = \frac{2L_{12}}{L_{11} - L_{22}} = \pm\frac{2\sqrt{r_0 r_2}}{r_1 - r_0 h_{22}} \cdot \frac{d\varphi}{d\alpha}. \tag{8}$$

Taking into account that

$$\frac{d\varphi}{d\alpha} = \tan\theta,$$

where θ is the angle between the direction of propagation of a wave in relation to the singular direction (α_0,φ_0), and choosing a positive sign in (8), yield

$$\psi_S = \frac{1}{2}\tan^{-1}(\xi\tan\theta), \tag{9}$$

with

$$\xi = \frac{2\sqrt{r_0 r_2}}{r_1 - r_0 h_{22}}.$$

Two sheets of the quasi-shear wave velocity surface in the $\pm 10^0$ neighbourhood of the singularity direction of $\alpha_0 = 59.8^0$, and $\varphi_0 = 0^0$ for an orthorhombic medium with $c_{11} = 9$, $c_{12} = 3.6$, $c_{13} = 2.25$, $c_{22} = 9.84$, $c_{23} = 2.4$, $c_{33} = 5.9375$, $c_{44} = 2$, $c_{55} = 1.6$, $c_{66} = 2.182$ are shown in Fig. 1. These parameters are used by many authors as a standard model. In the same figure, the cylindrical surface calculated using formula (7) is shown.

The polarization vectors of the two quasi-shear waves in the same neighbourhood are demonstrated in Figs. 2 and 3. Those vectors have been expanded into the eigenvectors of the matrix $\mathbf{G}(\alpha_0,\varphi_0)$. They turn about when the direction of wave propagation rounds the singularity.

Conclusion

1. The formulas for calculating the phase velocities of quasi-shear waves in the neighbourhood of a singular direction are derived.

2 Analysis of the formulas allow us to draw the following conclusion: It is impossible to choose directions of the polarization vector such that they be continuous every where on a closed curve containing a singularity.

References

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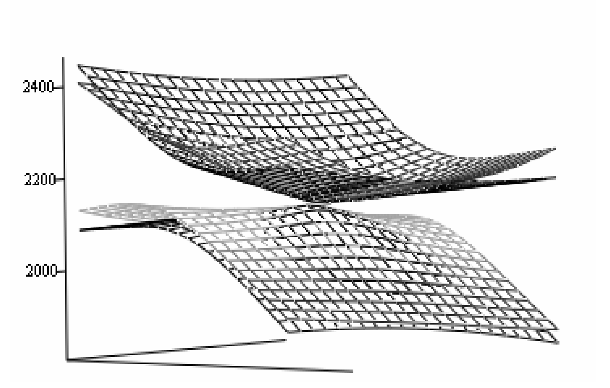

Figure.1. Intersection of the sheets of shear wave velocitiy surface and their approximation by a cylindrical surface

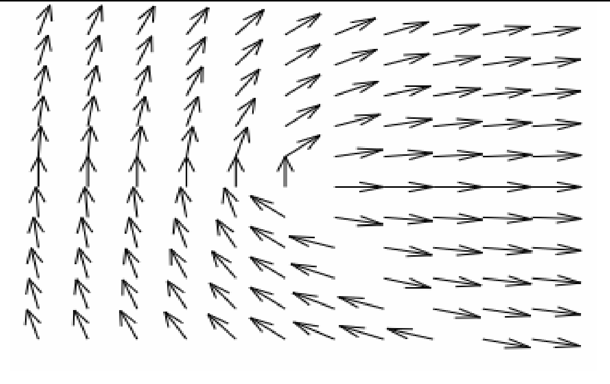

Figure 2 Disposition of the polarization vectors of the waves qS2 in the neighbourhood of the singular direction

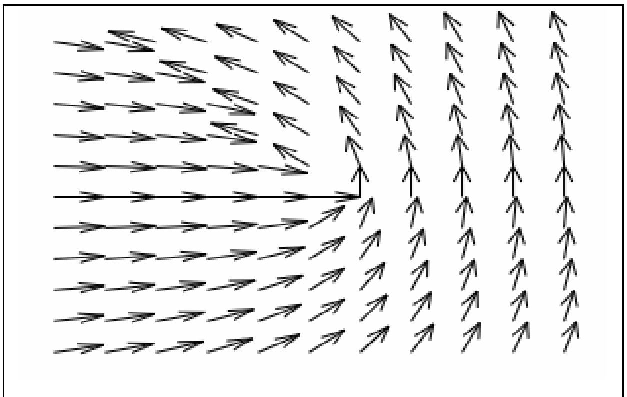

Figure 3 Disposition of the polarization vectors of the waves qS1 in the neighbourhood of a singular direction