P042

Low-frequency Wave Propagation in Periodically Layered Media

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SUMMARY

To compute the effective matrix from the stack of the layers we use the Baker-Campbell-Hausdorff (BCH) series. From the truncated BCH series we derive the velocity dispersion equation that correctly describes the wave propagation at low frequencies. The explicit equations derived for an acoustic medium, periodically layered medium, medium with monoclinic anisotropy and the vertical propagation case. The derived equations are tested on the two-layer periodically layered medium and on the real well-log data. The first-order correction term in the velocity dispersion equation results in more accurate phase velocity at low frequencies. This correction is an extension of the Backus averaging method and can be used for upscaling of the well-log data and seismic modeling.

Introduction

At low frequencies, when the wavelength is much larger than the period of the stack of layers, the multilayered medium has the properties of effective homogeneous anisotropic medium (Rytov, 1956; Backus, 1962; Schoenberg and Muir, 1989). When frequency increases, we observe the velocity dispersion (Helbig, 1984) and periodically located pass- and stop-bands with propagation and attenuation of the waves (Stovas and Ursin, 2007). That leads to the specific filtering of the wave (Braga and Herrmann, 1992) and results in the frequency-dependent caustics in the group domain (Roganov and Stovas, 2010). Taking into account that an effective matrix $\tilde{\mathbf{M}}(\omega)$ is given by the logarithm from the product of the exponent matrices, we can apply the Baker-Campbell-Hausdorff (BCH) series (Serre, 1965). In mathematics, the BCH series are used to construct the Lie groups for the Lie algebras (Bourbaki, 1971). Using this technique, we can derive the expressions for the terms of the Taylor series with respect to frequency ω for an effective matrix $\tilde{\mathbf{M}}(\omega)$. The zero-order term in this series gives the well-known Backus averaging (Backus, 1962). We derive the first- and second-order terms of this series, and also extend this technique to the medium with arbitrary number of layers. That results in the correction terms for velocity dispersion. We also show that the dispersion equation in such media is the even function of frequency. The theory developed in this paper can be used for an extension of the Backus averaging technique for the low frequency wave propagation. This is an important issue for matching of well-log data with seismic data, smoothing of the well-log data, upscaling of the well-log data and seismic modeling.

Theory

We consider wave propagation in the infinite periodically layered medium with N layers in a period. Let us denote z_j, $j = 1,...,N$ as the layer thickness and $H = \sum_{j=1}^{N} z_j$ as the overall layer thickness. The vector containing the stress-strain components $\mathbf{f}(z)$ defined for each layer, satisfies the differential equation,

$$\frac{d\mathbf{f}(z)}{dz} = i\omega \mathbf{M}_j(\omega)\mathbf{f}(z), \tag{1}$$

where $i = \sqrt{-1}$ and matrix $\mathbf{M}_j(\omega)$ is defined by the horizontal slowness and the type of the medium. The vectors $\mathbf{f}(0)$ and $\mathbf{f}(H)$ can be expressed by $\mathbf{f}(H) = \mathbf{P}(\omega)\mathbf{f}(0)$, where $\mathbf{P}(\omega) = \exp(i\omega z_N \mathbf{M}_N)...\exp(i\omega z_1 \mathbf{M}_1)$ is the matrix propagator for the stack of the layers that depends on frequency ω and horizontal slowness p. By assuming that the matrix $\mathbf{P}(\omega)$ can be transformed to a diagonal matrix, $\mathbf{P}(\omega) = \mathbf{E}^{-1}\,\mathrm{diag}(\exp(i\omega H q_m))\mathbf{E}$, where the matrix $\mathbf{E}$ composed from the eigen-vectors of matrix $\mathbf{P}(\omega)$. If all values of $\exp(i\omega H q_m)$ are different, we can define the matrix $\tilde{\mathbf{M}}(\omega)$ by following equation

$$\tilde{\mathbf{M}}(\omega) = \frac{1}{i\omega H}\log \mathbf{P}(\omega) = \mathbf{E}^{-1}\,\mathrm{diag}(q_m)\mathbf{E}. \tag{2}$$

The Baker-Campbell-Hausdorff series

From equation (2), it follows that the terms in Taylor series for matrix $\tilde{\mathbf{M}}(\omega)$ can be computed by using the Baker-Campbell-Hausdorff series (Serre, 1965) that can be generalized for arbitrary number of arguments.

Let us denote $\mathbf{A}_k = \sum_{j=1}^{N} z_j^k \mathbf{M}_j^k$. The BCH series is given by

$$H\tilde{\mathbf{M}}(\omega) = \tilde{\mathbf{M}}_0 + i\omega\tilde{\mathbf{M}}_1 - \omega^2\tilde{\mathbf{M}}_2 + o(\omega^3), \tag{3}$$

$$\tilde{\mathbf{M}}_0 = \mathbf{A}_1, \qquad \tilde{\mathbf{M}}_1 = \frac{1}{2}\sum_{N\ge i>j\ge 1} z_i z_j \left(\mathbf{M}_i\mathbf{M}_j - \mathbf{M}_j\mathbf{M}_i\right),$$

$$\tilde{\mathbf{M}}_2 = -\frac{1}{6}\mathbf{A}_1^3 + \frac{1}{4}\left(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1\right) - \frac{1}{3}\mathbf{A}_3 + \frac{1}{2}\sum_{m\ge i>j>k\ge 1} z_i z_j z_k \left(\mathbf{M}_i\mathbf{M}_j\mathbf{M}_k + \mathbf{M}_k\mathbf{M}_j\mathbf{M}_i\right). \tag{4}$$

By taking the limit for the number of layers and taking into account the continuity of matrices $\mathbf{M}_j(z)$, we can derive the equations for the gradient medium,

$$\tilde{\mathbf{M}}_0 = \mathbf{N}_1, \qquad \tilde{\mathbf{M}}_1 = \mathbf{N}_2 - \frac{1}{2}\mathbf{N}_1^2, \qquad \tilde{\mathbf{M}}_2 = -\frac{1}{6}\mathbf{N}_1^2 + \frac{1}{2}\mathbf{N}_3, \tag{5}$$

$$\mathbf{N}_1 = \int_0^H \mathbf{M}(z)\,dz, \quad \mathbf{N}_2 = \int_0^H \mathbf{M}(z_2)\,dz_2 \int_0^{z_2} \mathbf{M}(z_1)\,dz_1, \quad \mathbf{N}_3 = \int_0^H \mathbf{M}(z_3)\,dz_3 \int_0^{z_3} \mathbf{M}(z_2)\,dz_2 \int_0^{z_2} \mathbf{M}(z_1)\,dz_1. \tag{6}$$

For the vertical propagation and finite number of isotropic or transversely isotropic layers, the elements of matrices $\tilde{\mathbf{M}}_0$, $\tilde{\mathbf{M}}_1$ and $\tilde{\mathbf{M}}_2$ can be given by simple equations. For a single layer *j,* we have

$$z_j\mathbf{M}_j = t_j \begin{pmatrix} 0 & Z_j^{-1} \\ Z_j & 0 \end{pmatrix}, \tag{7}$$

where $Z_j = \rho_j V_j$ is the impedance, $t_j = z_j / V_j$ is the vertical traveltime in layer *j*, ρ_j is the density and V_j is the vertical velocity. From equations (4) we obtain

$$\tilde{\mathbf{M}}_0 = \begin{pmatrix} 0 & b_0 \\ a_0 & 0 \end{pmatrix}, \quad \tilde{\mathbf{M}}_1 = \begin{pmatrix} -a_1 & 0 \\ 0 & a_1 \end{pmatrix}, \quad \tilde{\mathbf{M}}_2 = \begin{pmatrix} 0 & b_2 \\ a_2 & 0 \end{pmatrix}, \tag{8}$$

$$a_0 = \sum_{j=1}^{N} t_j Z_j, \quad b_0 = \sum_{j=1}^{N} \frac{t_j}{Z_j}, \quad a_1 = \frac{1}{2}\sum_{N\ge i>j\ge 1} t_i t_j \left(\frac{Z_i}{Z_j} - \frac{Z_j}{Z_i}\right), \tag{9}$$

$$a_2 = -\frac{1}{6}a_1^2 b_1 + \frac{1}{2}a_1\sum_{j=1}^{N} t_j^2 - \frac{1}{3}\sum_{j=1}^{N} t_j^3 Z_j + \sum_{N\ge i_3>i_2>i_1\ge 1} t_{i_1}t_{i_2}t_{i_3}\frac{Z_{i_1}Z_{i_3}}{Z_{i_2}}$$

$$b_2 = -\frac{1}{6}a_1 b_1^2 + \frac{1}{2}b_1\sum_{j=1}^{N} t_j^2 - \frac{1}{3}\sum_{j=1}^{N} \frac{t_j^3}{Z_j} + \sum_{N\ge i_3>i_2>i_1\ge 1} t_{i_1}t_{i_2}t_{i_3}\frac{Z_{i_2}}{Z_{i_1}Z_{i_3}}, \tag{10}$$

For the gradient medium, it is convenient to introduce the vertical traveltime $\tau(z) = \int_0^z \frac{d\xi}{V(\xi)}$, and to use it as an integration variable. If we denote $T = \tau(H)$ for the ray traveltime through the period, equations (9)-(10) can also be used after substitution of all sums in equations above by integrals.
For vertical propagation in binary (two-layer) medium, the velocity dispersion equation takes the form

$$t^2 = \tau_0 + \omega^2\tau_2 + \omega^4\tau_4 + o\left(\omega^5\right), \tag{11}$$

where the coefficients

$$\tau_0 = t_1^2 + t_2^2 + t_1t_2\left(\frac{Z_1}{Z_2} + \frac{Z_2}{Z_1}\right), \quad \tau_2 = \frac{1}{12}t_1^2t_2^2\left(\frac{Z_1}{Z_2} - \frac{Z_2}{Z_1}\right)^2, \quad \tau_4 = \frac{\tau_2}{15}\left[\tau_0 + t_1t_2\left(\frac{Z_1}{Z_2} + \frac{Z_2}{Z_1}\right)\right]. \tag{12}$$

The dispersion equation (11) can also be defined in terms of reflection coefficient $r = (Z_2 - Z_1)/(Z_2 + Z_1)$ at interface between the layers,

$$\frac{1}{V^2(\omega)} = \left(\frac{\alpha_1}{V_1} + \frac{\alpha_2}{V_2}\right)^2 + \frac{4r^2}{1-r^2}\frac{\alpha_2\alpha_1}{V_2V_1} + \omega^2H^2\frac{4\alpha_1^2\alpha_2^2r^2}{3V_1^2V_2^2\left(1-r^2\right)^2}$$
$$+\,\omega^4H^4\frac{4\alpha_1^2\alpha_2^2r^2}{45V_1^2V_2^2\left(1-r^2\right)^2}\left[\left(\frac{\alpha_1}{V_1} + \frac{\alpha_2}{V_2}\right)^2 + \frac{2\alpha_1\alpha_2\left(1+3r^2\right)}{V_1V_2\left(1-r^2\right)}\right] + O\left(\omega^5\right). \tag{13}$$

The first term in (13) corresponds to the time-average (ray) velocity, that is the infinite-frequency limit. The sum of the first and the second terms represents the Backus (1962) velocity that is the zero-frequency limit. The terms at ω^2 and ω^4 are the first and the second correction terms, respectively.
If the layers possess the monoclinic anisotropy with horizontal plane of symmetry, the layer matrices can be partitioned into blocks, $\mathbf{M}_j = \begin{pmatrix} 0 & \mathbf{Y}_j \\ \mathbf{X}_j & 0 \end{pmatrix}$, where $\mathbf{X}_j, \mathbf{Y}_j \in GL_3(\mathbf{C})$ - are invertible 3x3 matrices.
In this case the first terms in the BCH series for a periodically layered medium are given by

$$\tilde{\mathbf{M}}_0 = \begin{pmatrix} 0 & \mathbf{B}_0 \\ \mathbf{A}_0 & 0 \end{pmatrix}, \qquad \tilde{\mathbf{M}}_1 = \begin{pmatrix} \mathbf{B}_1 & 0 \\ 0 & \mathbf{A}_1 \end{pmatrix}, \qquad \tilde{\mathbf{M}}_2 = \begin{pmatrix} 0 & \mathbf{B}_2 \\ \mathbf{A}_2 & 0 \end{pmatrix}, \tag{14}$$

where the matrices $\mathbf{A}_j$ and $\mathbf{B}_j$, $j = 0,\ldots,3$ are given by the weighted products of matrices $\mathbf{X}_k$ and $\mathbf{Y}_k$, $k = 1,2$,

$$\mathbf{A}_0 = \alpha_1\mathbf{X}_1 + \alpha_2\mathbf{X}_2, \quad \mathbf{B}_0 = \alpha_1\mathbf{Y}_1 + \alpha_2\mathbf{Y}_2, \quad \mathbf{A}_1 = \frac{\alpha_1\alpha_2}{2}\left(\mathbf{X}_2\mathbf{Y}_1 - \mathbf{X}_1\mathbf{Y}_2\right), \quad \mathbf{B}_1 = \frac{\alpha_1\alpha_2}{2}\left(\mathbf{Y}_2\mathbf{X}_1 - \mathbf{Y}_1\mathbf{X}_2\right), \tag{15}$$

$$\begin{aligned} \mathbf{A}_2 &= \alpha_1^2\alpha_2\left(\mathbf{X}_2\mathbf{Y}_1\mathbf{X}_1 + \mathbf{X}_1\mathbf{Y}_1\mathbf{X}_2 - 2\mathbf{X}_1\mathbf{Y}_2\mathbf{X}_1\right) + \alpha_1\alpha_2^2\left(\mathbf{X}_2\mathbf{Y}_2\mathbf{X}_1 + \mathbf{X}_1\mathbf{Y}_2\mathbf{X}_2 - 2\mathbf{X}_2\mathbf{Y}_1\mathbf{X}_2\right) \\ \mathbf{B}_2 &= \alpha_1^2\alpha_2\left(\mathbf{Y}_2\mathbf{X}_1\mathbf{Y}_1 + \mathbf{Y}_1\mathbf{X}_1\mathbf{Y}_2 - 2\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_1\right) + \alpha_1\alpha_2^2\left(\mathbf{Y}_2\mathbf{X}_2\mathbf{Y}_1 + \mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_2 - 2\mathbf{Y}_2\mathbf{X}_1\mathbf{Y}_2\right) \end{aligned}, \tag{6}$$

where $\alpha_j = z_j / H$, $j = 1,2$ is fraction ratio for *j*-th layer.

Numerical examples

In order to illustrate the velocity dispersion equation, we consider the periodically layered model consisting of two layers with the following properties, $V_{P1} = 2000\,m/s$, $V_{S1} = 700\,m/s$, $\rho_1 = 2000\,kg/m^3$, $V_{P2} = 3000\,m/s$, $V_{S2} = 900\,m/s$ and $\rho_2 = 2200\,kg/m^3$. The period thickness is $H = 40\,m$ and $\alpha_1 = \alpha_2 = 0.5$. In Figure 1 (left) we show the exact velocity, the time-average velocity, the Backus velocity and velocities with first-order correction term and both first- and second-order correction terms. One can see that including the correction terms improve the accuracy of velocity dispersion curve. The main contribution is related to the first-order term. The effect of all correction terms increases with increasing in reflection coefficient.
In Figure 1 (right) we illustrate the behavior of the phase velocity by considering the same model as above but non-vertical propagation. The exact velocity, the Backus velocity and the Backus velocity with first-order correction term computed for frequency of 10Hz are given versus horizontal slowness. The accuracy of the first-order correction is much better than for the Backus velocity.
We also apply the low frequency correction technique for the real well-log data from the North Sea. In Figure 2 we show the velocity profile and the smoothed by the Backus averaging (left) with smoothing window of 40m. The error in velocity computed for Backus averaging and Backus averaging with first-order correction is shown for 10Hz (in the middle) and 30 Hz (to the right). Generally, the all errors increase with frequency and the larger the more variable velocity-density profile within the smoothing window. Nevertheless, the Backus with first-order correction results in much smaller error than the pure Backus averaging.

Conclusions

To compute the effective matrix from the stack of the layers we use the BCH series. From the truncated BCH series we derive the velocity dispersion equation that correctly describes the wave propagation at low frequencies. The explicit equations derived for an acoustic medium, periodically layered medium, medium with monoclinic anisotropy and the vertical propagation case.
The derived equations are tested on the two-layer periodically layered medium and on the real well-log data. The first-order correction term in the velocity dispersion equation results in more accurate phase velocity at low frequencies. This correction is an extension of the Backus averaging method and can be used for upscaling of the well-log data and seismic modeling.

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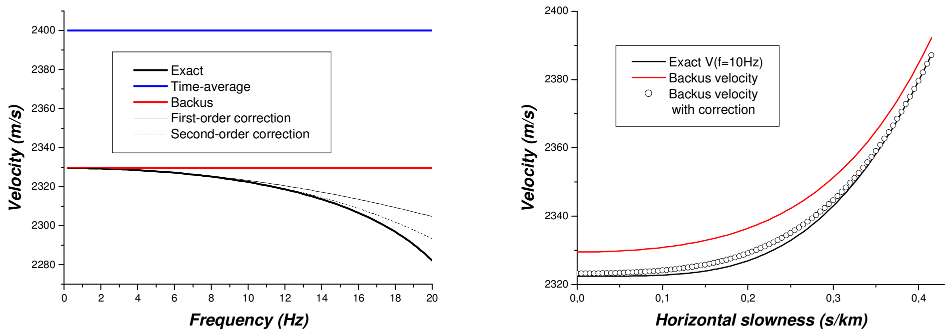

Figure 1 *Vertical velocities versus frequency for a periodically layered model (left). The oblique propagation (right). The effective phase velocity, the Backus velocity and the Backus velocity with correction versus horizontal slowness for 2-layer velocity model for frequency of 10Hz..*

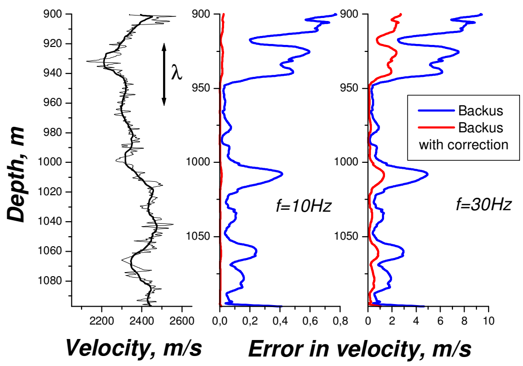

Figure 2 *Smoothing of the velocity from the well-log data (left) and errors in the Backus velocity shown by blue line and the Backus velocity with correction shown by red line for frequencies of 10 and 30 Hz.*