Wave Equation

Finite-difference Calculation in Tesseral

Content

1 Introduction

Wave propagation depends on the static *properties of background medium*. For seismic (sound) waves, these properties consist of:

> *P-wave velocity* ***Vp***, shear-wave velocity ***Vs*** *, density* ***ρ***. These properties may be used to describe ***macro-model*** (bulk model) of the medium.
>
> Other *medium* properties related to ***micro-model*** (subtle model) of the medium: *anisotropy* – usually described by *Thomsen parameters* ε, δ, γ and φ, α - angle of symmetry axis tilt and azimuth, *absorption* – usually described by ***Q****-factor, micro-fracturing, porosity and* etc;

Wave Propagation can be characterized using dynamic parameters, such as stresses (or pressures) and instantaneous particle velocity (displacement) – stress-displacement formulation.

Relationships between medium properties and dynamic parameters of wave propagation are defined by *differential equations*, which can be solved numerically to *simulate the real process* of wave propagation.

In numerical solution of *differential* wave *equations*, wave propagation is characterized as a *wavefield* ***W(p),*** which is a *vector function of 3D* space *(x, y, z)* and time *(t) variables*:

W(p)=**V**(x, y, z, t).

For numerical computation, *wave field* is characterized as a set of parameters which are usually distributed on *rectangular (orthogonal) grids* oriented along *X, Y, Z directions* in Cartesian coordinate system. Medium properties are also discretized into such kind of grid.

Wave equation and finite-difference modeling

Wave equation

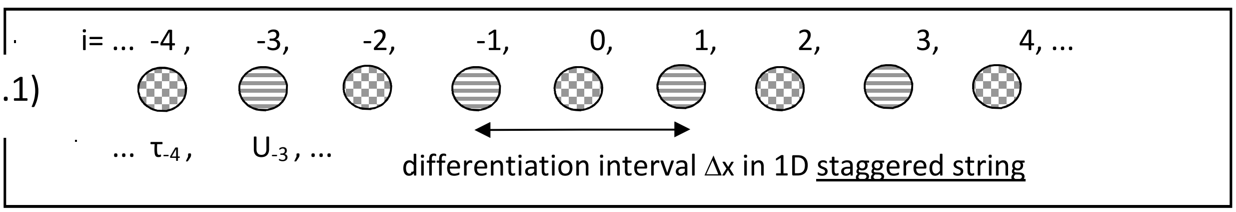
$$(1)\quad \frac{\partial^2 U(x,t)}{\partial t^2} = b(x)\frac{\partial}{\partial x}\left(\frac{\partial \tau(x,t)}{\partial t}\right)$$

is the differential equation for wave-propagation process, where x can be a scalar or vector value, e.g. (x_1, x_2, x_3) catertsian coordinates (usually denoted as x, y, z).

Let's consider 2 (differential) variables ***τ*** and ***U*** *(stress & displacement)* along the line with evenly-distributed discrete points ***i*** = ..., -2, -1, 0, 1, 2, ... Variables ***τ*** and ***U*** be allocated in *even* and *odd* discrete points, respectively.

Relationship between variables ***τ*** and ***U*** is also determined by the 1st-order differnetial equation (2), which can be solved using the staggered grid as shown below:

(1.1)



Δt – time step of computations k=0,1,2,...

$$(2)\qquad k = 0,1,2,\ldots \frac{[0]}{[1]}\begin{cases} i = \ldots,-2,0,2,\ldots & \left\{ \tau_i \leftarrow \tau_i + a_i \dfrac{\Delta t}{\Delta x}(U_{i+1} + U_{i-1}) \right. \\ i = \ldots,-1,1,\ldots & \left\{ U_i \leftarrow U_i + b_i \dfrac{\Delta t}{\Delta x}(\tau_{i+1} + \tau_{i-1}) \right. \end{cases}$$

The integral form of equation (2) can be written as:

$$(2.1)\quad \begin{aligned} \tau(x,t) &= a(x)\int_0^t \frac{\partial U(x,t)}{\partial x}\,dt \\ U(x,t) &= b(x)\int_0^t \frac{\partial \tau(x,t)}{\partial x}\,dt \\ &\text{where}\quad \Delta x,\ \Delta t \to 0 \end{aligned}$$

(2.1) is differentiated twice by ***t*****:**

$$(2.2)\quad \begin{aligned} \frac{\partial^2 \tau(x,t)}{\partial t^2} &= a(x)\frac{\partial}{\partial x}\left(\frac{\partial U(x,t)}{\partial t}\right) \\ \frac{\partial^2 U(x,t)}{\partial t^2} &= b(x)\frac{\partial}{\partial x}\left(\frac{\partial \tau(x,t)}{\partial t}\right) \end{aligned}$$

For each of variables **τ** and **U**, equation (2.2) corresponds to wave equation (1). And therefore solving equation (2) also corresponds to solving wave equation (1). Each time step of computations k=0, 1, 2, ... may be interpreted as process of wave propagation calculated in time increments. In terms of equation (2), each time step consists of ***two computation tacts***: 0- symmetry and 1-symmetry ([0] , [1]).^

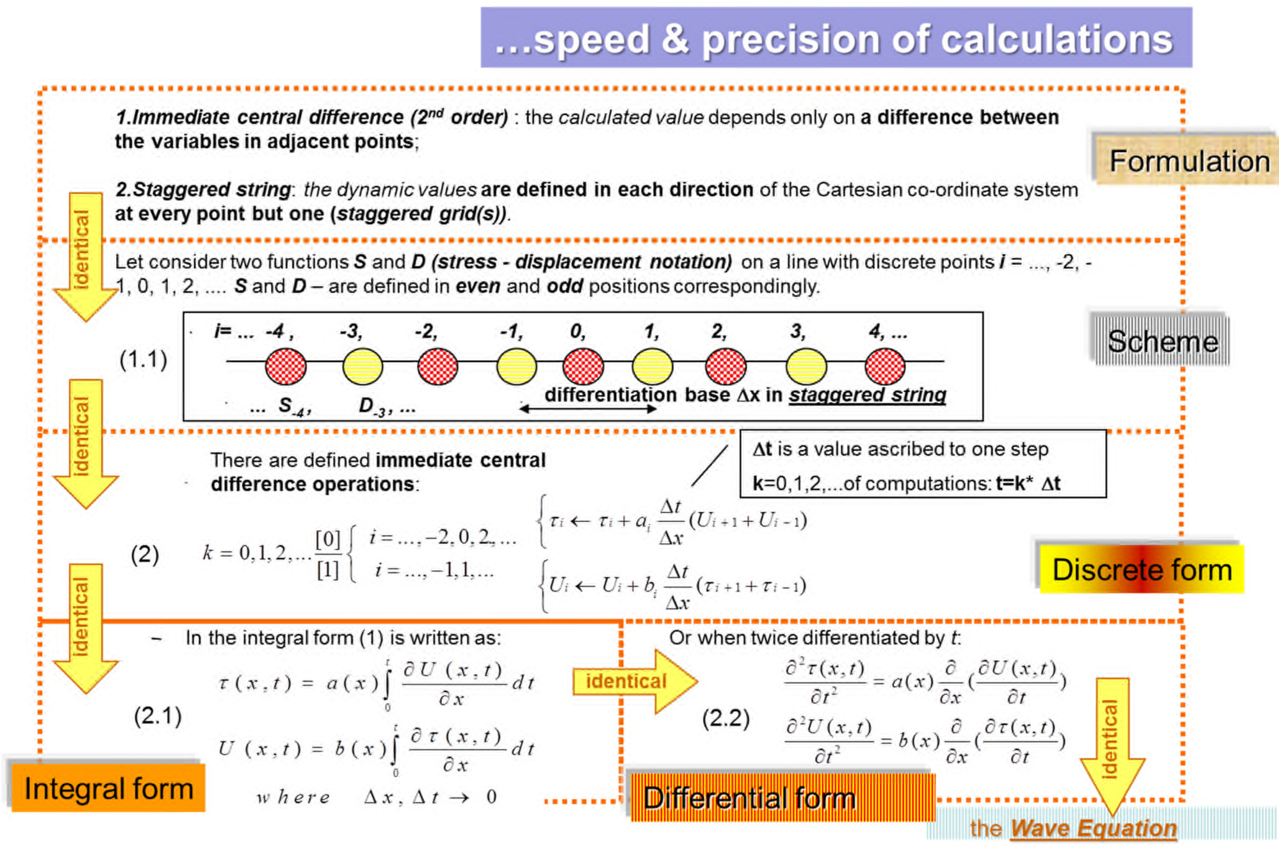
^Note: at numerical simulation of source point from which wavefield is starting propagating, formally it is assumed that ***1 tact= Δt/2***.

a_i and b_i are static medium *properties* distributed alnong spatial axis ***x***. If the differentiation base ***Δx*** is fixed (*grid)* and then time step ***Δt*** is also determined

Medium properties ***a_i and b_i*** can be scaled and used for computations on grid. Scaling constant ***G*** is a ***constant of the grid progression*** and has the unit as velocity (m/sec).

$$(3.1)\quad A_i = a_i \frac{\Delta t}{\Delta x} \quad or \quad A_i = \frac{a_i}{G}$$

$$(3.2)\quad B_i = b_i \frac{\Delta t}{\Delta x} \quad or \quad B_i = \frac{b_i}{G}, \quad where: G = \frac{\Delta x}{\Delta t}$$



This is a scheme of *finite-difference calculations of central type* (symmetric to point where variable is calculated) and *formally can be referred to (numerical) 2nd order approximations**. Equation (2.2) shows that solving equation (2) is asymptotically identical to solving wave equation (1) and accuracy of calculations is limited by the finiteness of *Δx* and *Δt* , and also by the precision of numerical representation and arithmetic operations for *τ* and *U*.

**** Note :*** formally, each of left- and right-side finite-difference approximation of differential operator is of 1st-order precision, central difference (2), which is their sum – is of 2nd order [2].

Equation (2) is the basis of the *staggered cell* (referred as *staggered grid* in literature [1]) *computation scheme* for finite-difference solving the wavefield propagation by discrete steps *Δt* on differentiation base *Δx*. On Fig. 2.1, the distributed cell is represented by 2 adjacent nodes. Because each of those 2 adjacent nodes corresponds to different properties and variables, the notion of *nodes symmetry* is introduced. Along the direction of differentiation (here by X) *even nodes (i=0, 2, ...) have symmetry [0] (τ and a), odd nodes (i=1,3, ...) have symmetry [1] (U and b)*. Distance between nodes is called as a *grid cell size* (here Δx/2). This discrete 1D representation of wave equation (2) can be extended to 2D and 3D cases.

There can be ***different approximations to wave equation*** depending on which physical properties of the medium are taken into account: ***scalar*** (P-velocity only) ***acoustic*** (P-velocity and density), ***elastic*** (additionally – S-velocities), ***anisotropic*** (additionally – Thompsen's parameters and up to 3 systems of micro-fracturing), ***visco-elastic*** (additionally to elastic – Q-factor).

References

1 Virieux, J., 1986, wave propagation in heterogeneous media: Velocity-stress finite-difference method: Geophysics, 51, 901.

2 Jaun A., 1999, Numerical methods for partial differential equations: Royal Institute of Technology, Stockholm.

2 Wavefield parameters

Seismic survey consists of seismic signal recorded by receivers for particular sources. Receiver and sources coordinates represent *survey geometry*.

Displacement (or instantaneous particle velocity) ***U*** is a vector (U_x ,U_y ,U_z) of particle movements as function of time ***t*** along each of axis X,Y, Z. For each *receiver (geophone)* position, variables $U_x(t)$, $U_y(t)$, U_z *(t)* represent the components (***C***) for each of 3 directions. In land surveys, maybe only component U_z is recorded (***1C*** observations) or all 3 components U_x ,U_y ,U_z is recorded (***3C*** – observations).

Stress (or pressure) τ is usually recorded in marine surveys by hydrophones.

The point where modeled wave start to propagate is interpreted as *the source generating the wave field*.

Depending on the no. of dimensions present in a wave-field computation, *wave field numerical approximation* can categorized into *1D, 2D* and *3D cases*. Each of 3 cases can have different appropriate applications to seismic data processing and interpretation.

3 *Spatial and temporal step in Tesseral* calculation

- ✓ Minimum number of differentiation bases per wavelength: ***BaseWave=10***
- ✓ Length of the computation grid differentiation base *Δd=(**Vp**[min]/**Ft**)/ **BaseWave;***
- ✓ *Cell size (default)* $\Delta x_{def}=\Delta d/2$ (1 cell=Δd/2 in the computation *staggered cell* used in Tesseral)
- ✓ *Time step (default) of computations*: $\Delta t_{def}=\Delta d\ /(Vp[max]*Stability)$

where:

Vp*[min]* – minimum compression wave velocity; in case of elastic (anisotropic, vicso-) approximation *Vp[min]* is divided by √2~=1.414;

Vp*[max]* – maximum compression wave velocity;

Ft – *Threshold (maximum) frequency* of the source wavelet;

Stability – stability constant = sqrt(2)+1/BaseWave ~ 1.5

- ✓ From (4): $\Delta d_{def}/\ \Delta t_{def}= Vp[max]*Stability$
  For actual (defined by user and/or by program) ***Δd*** and ***Δt*** must be observed:

  G must be ≤ ***Vp****[max]*****Stability***

  where: ***G=Δd/Δt -*** is a *constant of grid progression*;

4 *Tesseral* solutions for accelerating computation

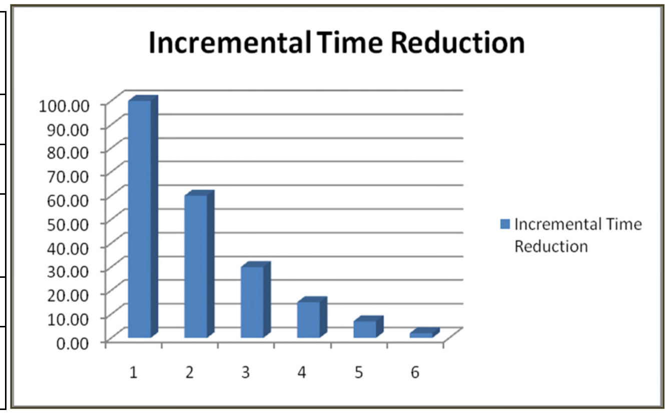
- ✓ Simple differentiation formula.**

**Note: from the staggered string property (1.1), it may be shown that *building higher-order central difference scheme may be less appropriate* due to asymptotic identity between formulae (1) and (2), as it is shown with (2.1) and (2.2).

- ✓ Because of the pattern of data distribution on the staggered computation grids, the data needed for calculations is packed much more densely.
- ✓ Absorption scheme for attenuating the artificial reflections from the grid borders (more than 99% of incoming energy to the border is absorbed) allows to have narrow absorbing margins and use much less computation resources for this purpose.
- ✓ Controlled calculation area which expands with the propagation of the waves.

Efficiency improvement in Tesseral releases

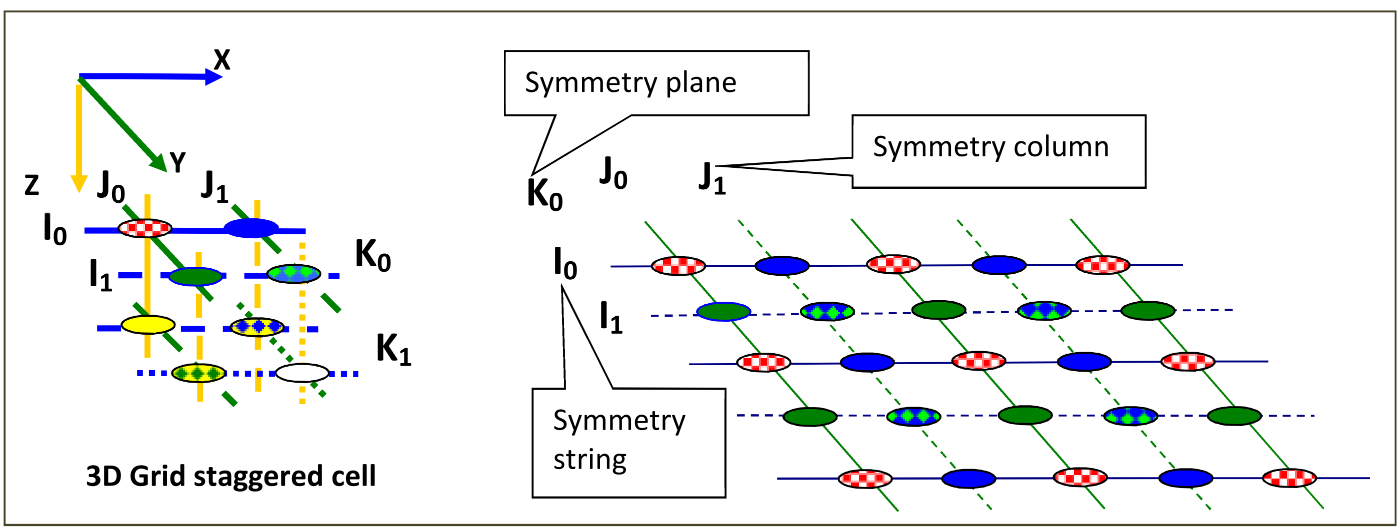
1	Benchmark model 3x3 km and 40 Hz signal frequency fofor 32-bit application;
2	Added one processor (parallelized x 2).
3	Increased word length (64-bit application).
4	Optimizations made for the internal loops and grid data distribution
5	Parallel computation on multi-core processors
6	Parallel computation on GPU-cards (x15-20 time reduction)



5 Differential equations of 2D wave motion in *Tesseral*

For the *stress-displacement* representation of wave equation, the *distributed cell* representation (discrete form) is as below.
For more theory, please also see [3 Wavefield Theory for Multiparameter Medium in Tesseral.pdf]

2D Calculations Plane in Tesseral



τ_{nm} – stress component ; $U_{1,2,...}$ – displacement (instantaneous particle velocity)

5.1 Calculations on Staggered Cell

Let consider 3D distributed cell in a Cartesian coordinates X,Y,Z (figure above):

In 2D and 3D numerical approximations of wave equation, variables ***τ*** and ***U*** are allocated onto particular nodes of distributed cell (grid) and participate in numerical (finite-difference) differentiation, in general case, along all directions (in 2D case - X,Y(Z), 3D case - X,Y,Z). From simple considerations such as extending the numerical method (2) to 2D and 3D cases, it follows that, in 3D case, variables ***τ*** and medium properties ***a*** must be allocated in nodes with symmetry: (0,0,0), (1,1,0), (1,0,1) and (0,1,1) (symmetries are notated by axis (x,y,z)). Variables ***U*** and (medium) properties **b** are allocated in nodes with symmetry (1,0,0), (0,1,0), (0,0,1) and (1,1,1).

In general, the properties ***a, b*** and the variables ***τ, U*** which are spaced in nodes of a distributed cell may be a ***vector***, i.e. consist of several member variables, each of which is involved in the differentiation operator along different directions in a different way. Such distributed vector along different coordinate directions are called ***tensors***.

If the properties (of simulated medium) in each node of a distributed cell consist of a single value (**scalar distribution**), this corresponds to the properties of ***isotropic medium*** (locally, the *velocity of the wave field propagation does not depend on the direction*).

If properties (of simulated medium) at some node (nodes) of distributed cell consist of vector values (***vector distribution***), this corresponds to the properties of ***anisotropic medium*** (*locally, the velocity of the wave field depends on its propagation direction*).

If a variable (usually ***U***) *in each node of the distributed cell* consist of *a single value*, in mathematical terms, it is called a ***scalar field***.

If a variable (typically ***τ***) at some node (nodes) of distributed *cell* consist of a *vector of values*, it is mathematically denoted as a ***vector field***.

Since the computational scheme of equation (2) simulates a physical process, it is important to comply with certain principles of invariance:

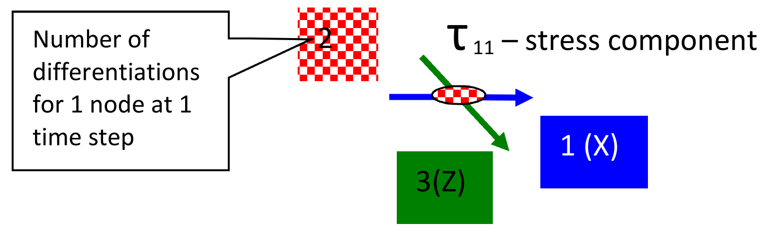
- **Invariance with respect to the order of differentiation by the coordinates**
  - Approximation (in this case) to the wave equation (continuation of the wave field in time), in the terms of physics, means also compliance with the law of energy conservation (for the process in time).
  - Transfer (across the grid) of stress ***τ*** and the displacement velocity ***U*** for some time *must be invariant to the time step of calculations* (*if we account for the stability of calculations, then starting with step less than or equal to* $\boldsymbol{\Delta t_{max}}$*). Otherwise the linearity with respect to time is violated, and accordingly, the law of conservation of energy is not complied with.*
  - *The expression "invariance with respect to the order of differentiation by the coordinates" (here) means that, within one time step of calculations, the results are invariant to the order of intermediate calculations. This invariance is achieved only if **τ** and **U** are calculated separately in a single tact for each time step.*
  - *If the results are invariant with respect to the order of the* intermediate calculations within a time step of computations and only (linearly) depend on the magnitude of this step, such calculations are invariant with respect to time step *Δt* for particular time length. That means that, in term of physics, the law of energy conservation is observed.

- **The invariance to the orientation of coordinate system**
  - The system of differential equations leading to a computation scheme of the distributed cell (type (2)) must remain invariant to the replacements within the coordinates system (indexes of variables).
  - Obtaining the system of differential equations with such invariant form in some cases may require the introduction (in the equation) of the additional equations and variable members. Such additional members shall be an independent from the numeric fields (i.e. not formally belong to the fields ***τ*** or ***U***).

5.2 Acoustic case

[0] symmetry tact

(1) $\dot{\tau}_{11} = a_{11}\left(\frac{\partial U_1}{\partial x_1} + \frac{\partial U_3}{\partial x_3}\right)$



[1] symmetry tact

(2) $\dot{U}_1 = b\frac{\partial \tau_{11}}{\partial x_1}$

(3) $\dot{U}_3 = b\frac{\partial \tau_{11}}{\partial x_3}$

where:

$a_{11} = \lambda + 2\mu = \rho\,\alpha^2$

is component of the stiffness tensor

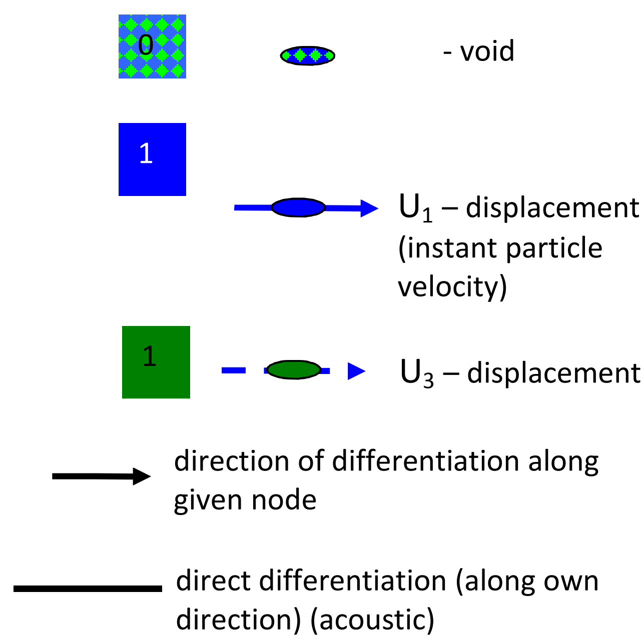
and

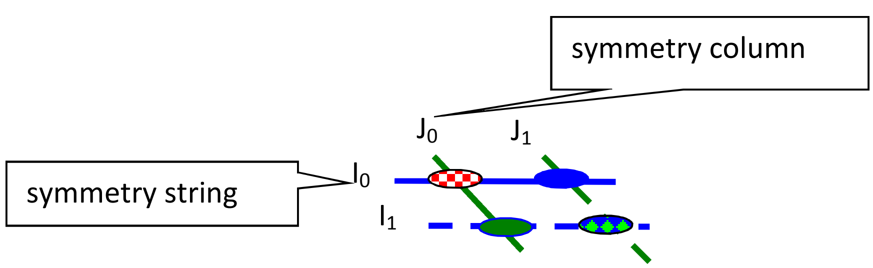
$\alpha =$ compression velocity

$b = \frac{1}{\rho}$

and

$\rho =$ density





2D Grid staggered cell

5.3 Elastic case

[0] symmetry tact

(1) $\dot{\tau}_{11} = a_{11}\dfrac{\partial U_1}{\partial x_1} + a_{13}\dfrac{\partial U_3}{\partial x_3}$

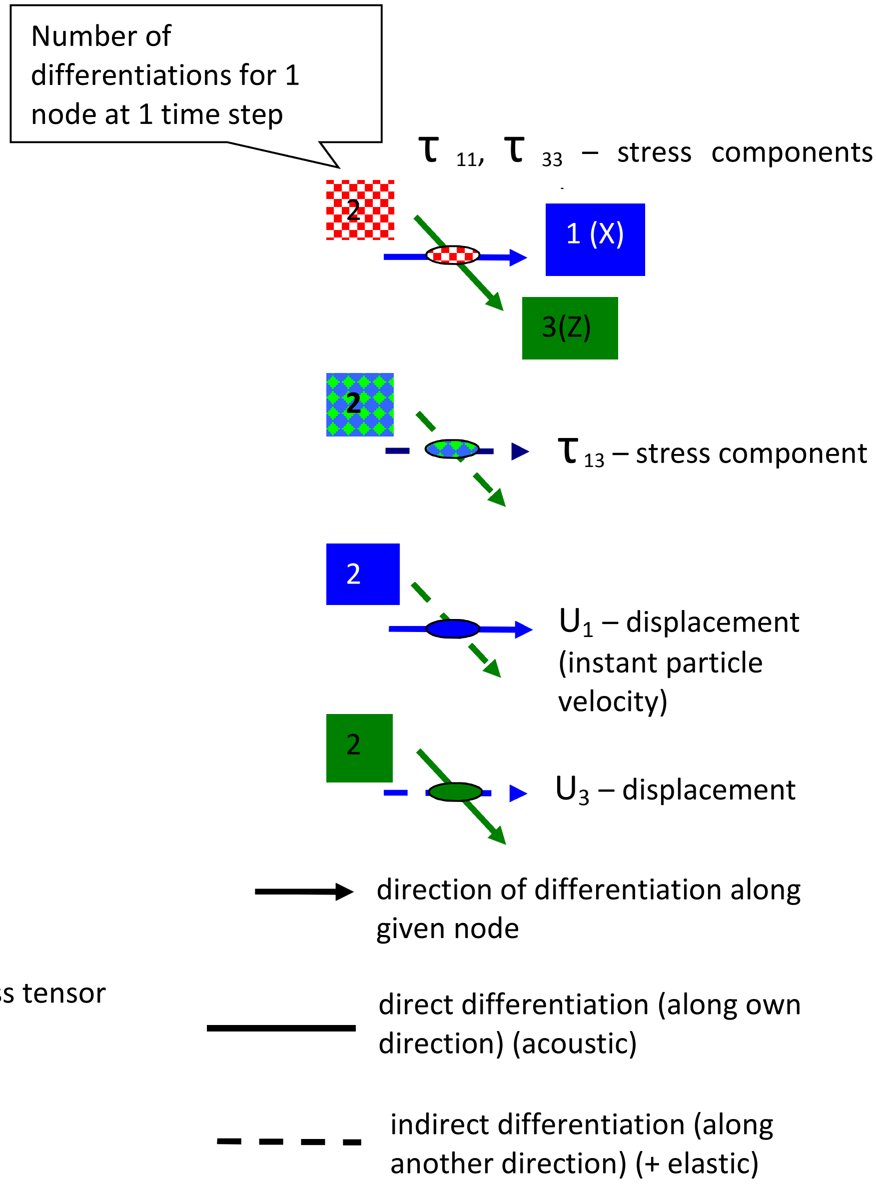
(2) $\dot{\tau}_{33} = a_{13}\dfrac{\partial U_1}{\partial x_1} + a_{33}\dfrac{\partial U_3}{\partial x_3}$

(3) $\dot{\tau}_{13} = a_{55}\left(\dfrac{\partial U_1}{\partial x_3} + \dfrac{\partial U_3}{\partial x_1}\right)$

[1] symmetry tact

(4) $\dot{U}_1 = b\left(\dfrac{\partial \tau_{11}}{\partial x_1} + \dfrac{\partial \tau_{13}}{\partial x_3}\right)$

(5) $\dot{U}_3 = b\left(\dfrac{\partial \tau_{3}}{\partial x_1} + \dfrac{\partial \tau_{33}}{\partial x_3}\right)$



where:

$a_{11} = a_{33} = \lambda + 2\mu = \rho\,\alpha^2$

$a_{55} = \mu = \rho\,\beta^2$

$a_{13} = \lambda = a_{11} - 2a_{55}$

a_{nm} – components of the stiffness tensor

and

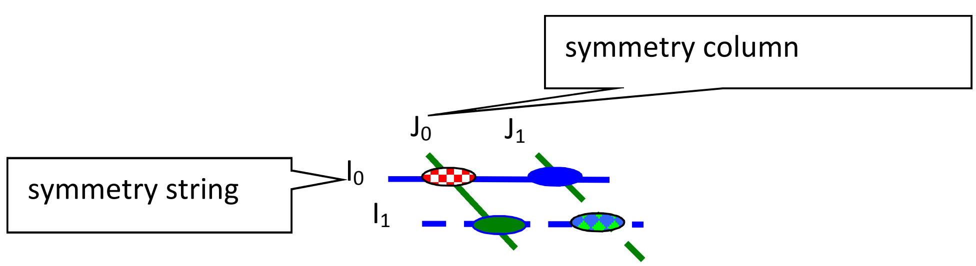
$\alpha =$ compression velocity

$\beta =$ shear velocity

$b = \dfrac{1}{\rho}$

and

$\rho =$ density



2D Grid staggered cell